Analysis IV

(for EL, GM, MX)

Week 3 - 03.03.25

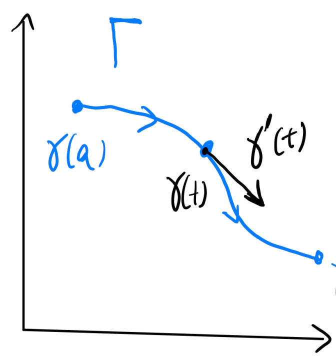
1. Complex integration

(Chapter 10 in the book)

Recall from vector analysis: If $\vec{F}$ is a <u>continuous</u> vector field on $\mathbb{R}^2$, so $\vec{F}: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$,

$$F(x_1, x_2) = \big(F_1(x_1, x_2), F_2(x_1, x_2)\big)$$

and $\gamma: [a, b] \longrightarrow \mathbb{R}^2$ is a C^1 curve with image Γ,



$$\int_\Gamma \vec{F}\, dl := \int_a^b \underbrace{\vec{F}(\gamma(t)) \cdot \vec{\gamma}'(t)}_{\text{inner product:}}\, dt$$

$$F_1(\gamma(t)) \cdot \gamma_1'(t) + F_2(\gamma(t))\, \gamma_2'(t)$$

(Orientation will be important!)

Recall:

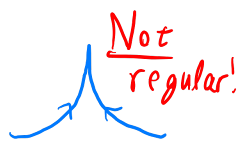
- The integral is independent of the parametrization, but if we switch orientation then it has the opposite sign!

  So if $h : [c,d] \to [a,b]$ is 1-1, onto and increasing, then γ and $\gamma \circ h$ are parametrizations of Γ with the same orientation.

- If the curve is <u>closed</u> (i.e. $\gamma(a) = \gamma(b)$) and $\operatorname{curl} F = 0$ in the interior of γ: $\boxed{\int_\Gamma \vec{F}\, d\ell = 0}$

- A curve is <u>simple</u>: If no self intersections.

- <u>Regular</u>: If $\gamma' \neq 0$.



We will define similarly the integrals of complex functions $f : \underset{\mathbb{R}^2}{\overset{\shortparallel}{\mathbb{C}}} \longrightarrow \underset{\mathbb{R}^2}{\overset{\shortparallel}{\mathbb{C}}}$

Definition: Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a regular curve that parametrizes $\Gamma \subseteq \mathbb{C}$.

Let $f : \Gamma \longrightarrow \mathbb{C}$ be a continuous function. Then

$$\int_\Gamma f(z)\, dz = \int_\gamma f(z)\, dz := \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt$$

Slight abuse of notation

Complex multiplication!

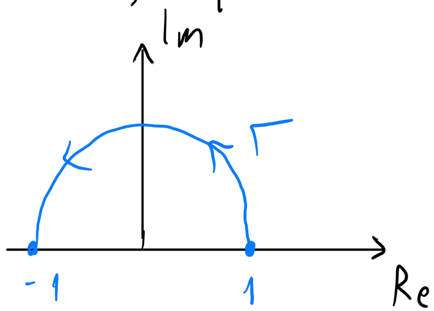
- Usual notation: $-\Gamma$ is Γ with the "opposite" parametrization

$$\int_{-\Gamma} f(z)\, dz = -\int_\Gamma f(z)\, dz$$

- The value of the integral is independent of the parametrization, as long as the orientation is the same.

Example 1:

Let Γ be the unit half-circle around $z=0$, parametrized anti-clockwise



$\gamma : [0,\pi] \longrightarrow \mathbb{C}$,

$\gamma(t) = e^{it}$

Then $\gamma'(t) = i e^{it}$.

Integrate $f(z) = z^2$ over Γ:

$$\int_\Gamma z^2 dz = \int_0^\pi (e^{it})^2 \cdot i e^{it} dt =$$

$$= i \int_0^{\pi} e^{3it}\, dt =$$

$$= i \left[\frac{e^{3it}}{3i} \right]_{t=0}^{t=\pi} = i \, \frac{e^{3i\pi} - e^{0}}{3i}$$

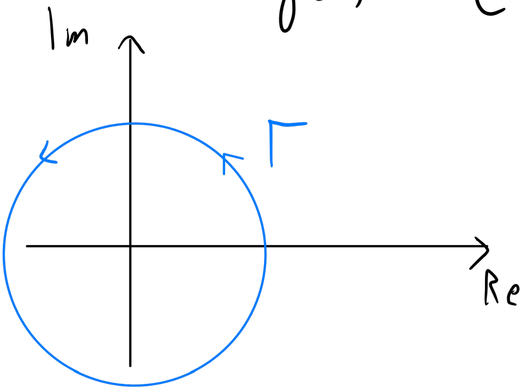
$$= -\frac{2}{3}.$$

In the above: We used the fundamental theorem of calculus and the fact that $e^{\lambda t} = \frac{1}{\lambda}\left(e^{\lambda t}\right)'$ for $\lambda \in \mathbb{C}$. For more general functions, we can always split in real and imaginary parts and compute the two (real) integrals separately.

__Example 2:__ $\Gamma = \{z : |z| = 1\}$, counter-clockwise.

$\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$,

$\gamma(t) = e^{it}$, $\gamma'(t) = ie^{it}$



$$\int_\Gamma z^2 \, dz = \int_0^{2\pi} (e^{it})^2 \cdot i e^{it} \, dt$$

$$= i \int_0^{2\pi} e^{3it} \, dt$$

$$= i \left[\frac{e^{3it}}{3i} \right]_{t=0}^{t=2\pi} = 0$$

We will see: Integrals of __holomorphic__ functions are governed by similar rules as integrals of vector fields with curl $= 0$.

Example 3: Same Γ and γ,

but $f(z) = \frac{1}{z}$.

$$\int_\Gamma \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{e^{it}}\, i e^{it}\,dt = i\int_0^{2\pi} dt = 2\pi i \neq 0.$$

Example 4: Same Γ (and γ),

$$f(z) = \frac{1}{z^2}.$$

$$\int_\Gamma \frac{1}{z^2}\,dz = \int_0^{2\pi} \frac{1}{(e^{it})^2}\, i e^{it}\,dt$$

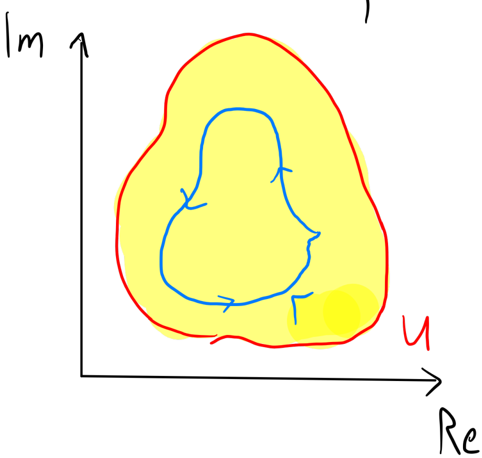
$$= i\int_0^{2\pi} e^{-it}\,dt = i\left[\frac{e^{-it}}{-i}\right]_{t=0}^{2\pi} = 0.$$

2. Cauchy's theorem

Let $U \subseteq \mathbb{C}$ be an open and simply connected set and let Γ be a closed and piecewise regular curve inside U.

Let $f : U \longrightarrow \mathbb{C}$ be holomorphic.

Then $$\int_\Gamma f(z)\, dz = 0.$$



Remarks:

1) Simply connected:

Means only one component and has no "holes".

Example: $\mathbb{C}$: Simply connected

$\{z : |z| < 1\}$: Simply connected.

$\mathbb{C} \setminus \{0\}$: NOT simply connected.

2) In practice: We are given f and Γ, and we try to find a simply connected set U which contains Γ and lies inside the domain where f is holomorphic

3) Piecewise regular: Might have corners.

Outline of the proof:

Let $\gamma(t) = \alpha(t) + \beta(t)i$ parametrize Γ, and

$$f(x+yi) = u(x,y) + v(x,y)i.$$

Then:

$$\int_\Gamma f(z)\,dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,dt =$$

$$= \int_a^b \big(u(\gamma(t)) + v(\gamma(t))\cdot i\big)\big(\alpha'(t) + \beta'(t)i\big)\,dt$$

$$= \underbrace{\left(\int_a^b \big(u(\gamma(t))\cdot \alpha'(t) - v(\gamma(t))\cdot \beta'(t)\big)\,dt\right)}_{\text{Real part}} +$$

$$+\left(\underbrace{\int_a^b \left(u(\gamma(t))\cdot\beta'(t)+v(\gamma(t))\cdot\alpha'(t)\right)dt}_{\text{Imaginary part}}\right)i$$

$$=\int_a^b \underbrace{\begin{pmatrix}u(\gamma(t))\\-v(\gamma(t))\end{pmatrix}}_{\vec{F}(\gamma(t))}\cdot\underbrace{\begin{pmatrix}\alpha'(t)\\\beta'(t)\end{pmatrix}}_{\gamma'(t)}dt+i\int_a^b \underbrace{\begin{pmatrix}v(\gamma(t))\\u(\gamma(t))\end{pmatrix}}_{\vec{G}(\gamma(t))}\cdot\underbrace{\begin{pmatrix}\alpha'(t)\\\beta'(t)\end{pmatrix}}_{\gamma'(t)}dt$$

$$=\int_\Gamma \vec{F}\,dl+\left(\int_\Gamma \vec{G}\,dl\right)i$$

Applying Green's theorem

$$=\int_{\text{int}\,\Gamma}\text{curl}\,F\,dx\,dy+\left(\int_{\text{int}\,\Gamma}\text{curl}\,G\,dx\,dy\right)i \quad (1)$$

But $\operatorname{curl} F = \partial_1 F_2 - \partial_2 F_1 = -\partial_x v - \partial_y u$

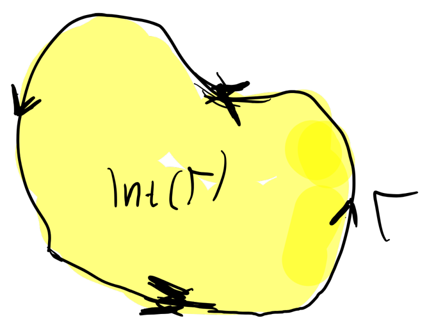
$\operatorname{curl} G = \partial_1 G_2 - \partial_2 G_1 = \partial_x u - \partial_y v$

By Cauchy-Riemann: $\operatorname{curl} F = 0 = \operatorname{curl} G$

So ① $= 0$. $\square$

Here we used that since U is simply connected and $\Gamma \subseteq U$, we have $\operatorname{int}\Gamma \subseteq U$, so F, G are defined on $\operatorname{int}\Gamma$.

Recall: $\operatorname{int}\Gamma$ = "interior of Γ".



Example: $\Gamma = \{z : |z| = 1\}$, $f(z) = z^2$

Applying Cauchy's theorem for $U = \mathbb{C}$:

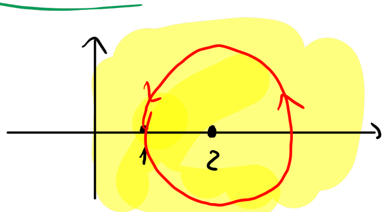
$$\int_\Gamma f(z)\, dz = 0$$

Example: For Γ as above, if $f(z) = \frac{1}{z}$:

f is holomorphic on $\mathbb{C} \setminus \{0\}$, there is no simply connected $U \subseteq \mathbb{C} \setminus \{0\}$ which contains Γ, so Cauchy's theorem does not apply.

(recall that we computed $\int_\Gamma f(z) dz \neq 0$).

Example: $f(z) = \frac{1}{z}$, but $\Gamma = \{z : |z-2| = 1\}$



In this case:

If $U = \{z : \operatorname{Re} z > 0\}$, it is simply connected and contains Γ. So

$$\int_\Gamma f(z)\, dz = 0.$$

But let's try to verify it explicitly:

$\gamma(t) = 2 + e^{it}$, $\gamma'(t) = ie^{it}$

$$\int_\Gamma f(z)\, dz = \int_0^{2\pi} \frac{1}{2+e^{it}} \cdot ie^{it}\, dt$$

$$= \int_0^{2\pi} \left(\log(2+e^{it})\right)' dt$$

$$= \left[\log\left(2+e^{it}\right)\right]_{t=0}^{t=2\pi} = 0$$

Here: We use the fact that the argument of log does not go through the negative real line, so $\log(z)$ remains differentiable along the curve with $(\log z)' = \frac{1}{z}$.

3. An extension of Cauchy's theorem.

Let Γ_0, Γ_1 be closed, simple, regular curves with

$$\Gamma_1 \subseteq \operatorname{int} \Gamma_0$$

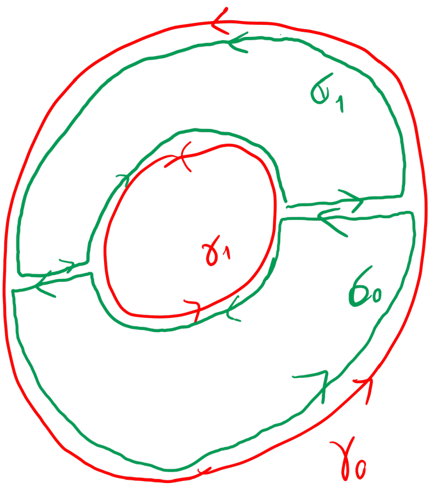
Let γ_0, γ_1 be parametrizations of Γ_0, Γ_1

with the same orientation.

If f is holomorphic on $\operatorname{int}\Gamma_0 / \operatorname{int}\Gamma_1$ (so "between the curves") then

$$\int_{\Gamma_0} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz.$$

Idea of proof:



Consider curves σ_0 and σ_1 as in the picture. Since f is holomorphic in the interior of both σ_0 and σ_1:

$$\int_{\sigma_0} f(z)\,dz = 0 = \int_{\sigma_1} f(z)\,dz.$$

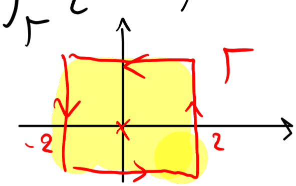
So $$0 = \int_{\sigma_0} f(z)\,dz + \int_{\sigma_1} f(z)\,dz$$

$$= \int_{\gamma_0} f(z)\,dz - \int_{\gamma_1} f(z)\,dz$$

since if I look at $\sigma_0 \cup \sigma_1$:
It covers γ_0 with the same orientation, γ_1 with the opposite orientation and the integrals over the segments cancel out. $\blacksquare$

Very useful result: Can be used to shift the curve of integration and obtain a simpler integral.

Example: Compute $\int_\Gamma \frac{1}{z}\,dz$, Γ: the square



Since $\frac{1}{z}$ is holomorphic on $\mathbb{C}/\{0\}$:

Same value as for $\int_{\Gamma'} \frac{1}{z}\,dz$,

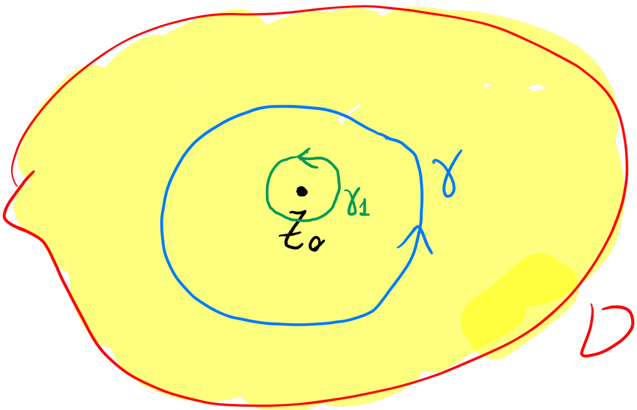
$\Gamma' = \{z : |z| = 1\}$, which we computed earlier. $(= 2\pi i)$

4. Cauchy integral Formulas

Suppose that $D \subseteq \mathbb{C}$ is open and $f : D \longrightarrow \mathbb{C}$ is holomorphic. If $z_0 \in D$, then

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0}\,dz$$

For any <u>closed</u>, simple curve γ inside D which contains z_0 in its interior and is counter-clockwise oriented.



Idea of proof:

$\dfrac{f(z)}{z-z_0}$ is holomorphic in $D \setminus \{z_0\}$.

By the application of Cauchy's theorem:

$\displaystyle\int_\gamma \frac{f(z)}{z-z_0}\,dz = \int_{\gamma_1} \frac{f(z)}{z-z_0}\,dz$ for γ_1 being any small circle around z_0 (as in the picture)

$$\gamma_1 : [0, 2\pi] \longrightarrow \mathbb{C}, \quad \gamma_1(t) = z_0 + r \cdot e^{it},$$

$r > 0$ small.

So: $$\int_{\gamma} \frac{f(z)}{z-z_0}\,dz = \int_{\gamma_1} \frac{f(z)}{z-z_0}\,dz$$

$$= \int_0^{2\pi} \frac{f(z_0+re^{it})}{\cancel{r}\cdot\cancel{e^{it}}}\, i\,\cancel{r}\cancel{e}^{it}\,dt$$

$$= i\int_0^{2\pi} f(z_0+r\cdot e^{it})\,dt$$

$$\xrightarrow{r\to 0} 2\pi i\, f(z_0).$$

(since f is continuous at z_0)

$\square$